

# ON THE FIXED POINTS OF THE MAP $x \mapsto x^x$ MODULO A PRIME

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**ABSTRACT.** In this paper, we show that for almost all primes  $p$  there is an integer solution  $x \in [2, p-1]$  to the congruence  $x^x \equiv x \pmod{p}$ . The solutions can be interpreted as fixed points of the map  $x \mapsto x^x \pmod{p}$ , and we study numerically and discuss some unexpected properties of the dynamical system associated with this map.

## 1. INTRODUCTION

**1.1. Motivation.** For a prime  $p$ , we consider the properties of the map

$$\psi_p : x \mapsto x^x \pmod{p}$$

when it acts on the integers  $x \in [1, p-1]$ . By the results Crocker [5] and Somer [18], there are at least  $\left\lfloor \sqrt{(p-1)/2} \right\rfloor$  and at most  $3p/4 + O(p^{1/2+o(1)})$ , respectively, distinct values of  $x^x \pmod{p}$  when  $1 \leq x \leq p-1$ .

We also note that various estimates depending on the multiplicative order modulo  $p$  of  $a$  on the number  $T(p, a)$  of solutions of the congruence

$$(1) \quad x^x \equiv a \pmod{p}, \quad 1 \leq x \leq p-1,$$

have been given in [1, 2]. In the most favorable case of  $a = 1$ , by [1, Corollary 5], we have

$$(2) \quad T(p, 1) \leq p^{1/3+o(1)}$$

as  $p \rightarrow \infty$ . Furthermore, by [1, Bound (2)], for any integer  $a$  we have  $T(p, a) \leq p^{11/12+o(1)}$ . Moreover, it is also shown in [1, Theorem 8] that the estimate

$$\#\{1 \leq x, y \leq p-1 : x^x \equiv y^y \pmod{p}\} \leq p^{48/25+o(1)}$$

holds as  $p \rightarrow \infty$ .

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The map  $\psi_p$  also appears in some cryptographic protocols (see [12, Sections 11.70 and 11.71]), so it certainly deserves more attention. Several conjectures and numerical data concerning this map can be found in [8].

Here, we address an apparently new problem and study the fixed points of the map  $\psi_p$ . Let  $F(p)$  denote the number of fixed points of the map  $\psi_p$ . That is,

$$F(p) = \#\{1 \leq x \leq p-1 : x^x \equiv x \pmod{p}\}.$$

Obviously  $x = 1$  is always a fixed point, which we call *trivial*. We show that for most primes  $p$  the map  $\psi_p$  has a *nontrivial* fixed point  $x \in [2, p-1]$ . Thus, we are interested in primes  $p$  with  $F(p) > 1$ . In the opposite direction, it has been noted in [1, Theorem 8] that the method used to prove (2) also applies to the congruence  $x^{x-1} \equiv 1 \pmod{p}$ , and thus it implies the bound

$$(3) \quad F(p) \leq p^{1/3+o(1)}$$

as  $p \rightarrow \infty$ .

We also study the quantity  $F(p)$  and other dynamical properties (such as the period statistics) of the map  $\psi_p$  numerically. In particular, these numerical results reveal that a naïve point of view of treating  $\psi_p$  as a “random” function on the set  $\{1, \dots, p-1\}$  is totally wrong. In particular, the numerical results significantly deviate from those predicted for truly random maps by the work of Flajolet and Odlyzko [6]. These results indicate that  $\psi_p$  tends to have shorter orbits and more fixed points than a random map even after removing the trivial fixed point  $x = 1$ . On the other hand, it is highly likely that the bound (3) is very far from being tight. We give some partial explanation for the “non-randomness” phenomenon, and introduce the notion of *random endomorphisms* in groups, which allows us to give some qualitative explanation for the numerical results. We consider developing a rigorous analysis of the random endomorphisms to be a challenging and important open topic.

Finally, in Section 5, we study the map  $x \rightarrow x^{f(x)} \pmod{p}$  for general polynomials  $f(X) \in \mathbb{Z}[X]$ , and show that such a map can have at most  $p^{6/13+o(1)}$  fixed points, as  $p \rightarrow \infty$ .

**1.2. Notation.** Before we give the precise statement we introduce some notation.

We define  $\log x$  as  $\log x = \max\{\ln x, 2\}$  where  $\ln x$  is the natural logarithm. Furthermore, for an integer  $k \geq 2$ , we define recursively  $\log_k x = \log(\log_{k-1} x)$ .

Throughout the paper, we use the Landau symbols  $O$  and  $o$  and the Vinogradov symbols  $\gg$  and  $\ll$  with their usual meanings. We recall that  $A = O(B)$ ,  $A \ll B$  and  $B \gg A$  are all equivalent to the fact that  $|A| < cB$  holds with some constant  $c$ , while  $A = o(B)$  means that  $A/B \rightarrow 0$ .

We further define the logarithmic integral  $\text{li}(N) := \int_2^N \frac{dt}{\log t}$ .

We always use  $p$  and  $q$  for prime numbers. We also use  $\varphi(k)$  and  $\omega(k)$  to denote the Euler function and the number of distinct prime divisors of an integer  $k$ .

Furthermore,  $\mathbb{F}_p$  denotes a finite field of  $p$  elements, which we consider to be represented by the elements of the set  $\{0, \dots, p-1\}$ , while  $\mathbb{Z}/q\mathbb{Z}$  denotes the residue ring modulo an integer  $q \geq 1$ .

**1.3. Heuristics on primes without nontrivial fix points.** Let us write  $\mathcal{A}$  for the set of prime numbers  $p$  for which  $\psi_p$  does not have a nontrivial fixed point  $x \in [2, p-1]$ :

$$\mathcal{A} = \{p \text{ prime} : F(p) = 1\}.$$

One easily finds that  $\mathcal{A}$  is not empty. In particular, among the first 1000 primes, there are precisely 72 of them in  $\mathcal{A}$ . The first few elements of  $\mathcal{A}$  are

$$(4) \quad 3, 5, 7, 11, 53, 59, 83, 107, 179, 227, 269, \dots$$

Quite likely, the set  $\mathcal{A}$  is infinite, but we have not been able to prove this unconditionally. However, we can show this under some standard conjectures about prime numbers. For example, assume that

$$(5) \quad p \equiv 3 \pmod{8} \quad \text{and} \quad p-1 = 2q,$$

where  $q$  is prime (several elements from the above list (4): 11, 59, 83, 107, 179, 227, are of this form). Consider an integer solution  $x$  to  $x^{x-1} \equiv 1 \pmod{p}$ . Then the multiplicative order of  $x$  divides  $x-1$ , which is an integer less than  $p-1$ . However, this multiplicative order must also divide  $p-1 = 2q$ . So, the only possibilities are that the order of  $x$  is either 2 or  $q$ . If it is 2, then  $x = 1$  (which is excluded) or  $x = p-1$ , which is not a fixed point as  $\psi_p(p-1) = 1$ . If it is  $q$ , then  $q \mid x-1$ , and since  $x-1 < 2q$ , we get that  $x-1 = q$ , so  $x = q+1 = (p+1)/2$ . Thus, we arrive at

$$1 \equiv x^{x-1} \equiv ((p+1)/2)^{(p-1)/2} \equiv 2^{-(p-1)/2} \equiv 2^{(p-1)/2} \pmod{p},$$

by Fermat's Little Theorem, which, in particular, implies that 2 is a quadratic residue modulo  $p$ . But this is impossible as  $p \equiv 3 \pmod{8}$ .

Standard conjectures then suggest that  $\mathcal{A}$  is infinite, and, in fact, putting

$$\mathcal{A}(N) = \mathcal{A} \cap [1, N],$$

the standard heuristic on the density of primes  $p$  satisfying (5) makes us conjecture that the inequality

$$\#\mathcal{A}(N) > c_0 N / (\log N)^2$$

holds for all  $N \geq 2$  with some positive constant  $c_0$ .

In Section 3, we give some further heuristic arguments suggesting that the stronger inequality

$$(6) \quad \#\mathcal{A}(N) \geq \frac{N}{(\log N)^2} \exp((1/\ln 2 + o(1)) \log_3 N \log_4 N)$$

holds as  $N \rightarrow \infty$ . In fact, in Section 3.1 we also give a heuristic argument that the “likelihood” of  $\psi_p$  having no nontrivial fix points is of order  $\exp(-\gamma(p) \cdot \tau(p-1))$ , where  $\tau(p-1)$  denotes the number of divisors of  $p-1$  and  $\gamma(p)$  is some explicit but quite irregular function of  $p$  taking values in  $(0, 1)$ ; see (20) for more details. In particular, we expect that  $\psi_p$  is very likely to have nontrivial fixed points unless the number of prime factors of  $p-1$  is very small.

**1.4. Main result.** We obtain an unconditional result in the opposite direction of the previous heuristics, in the sense that  $\mathcal{A}$  is fairly sparse. In particular, the estimate  $\#\mathcal{A}(N) = o(\pi(N))$  holds as  $N \rightarrow \infty$ , where, as usual, for a positive real number  $x$  we use  $\pi(x)$  to denote the number of primes  $p \leq x$ .

Let

$$(7) \quad \vartheta = \frac{1}{\zeta(2)} - \frac{1}{2\zeta(2)^2} = \frac{6\pi^2 - 18}{\pi^4} \simeq 0.4231 \dots,$$

where  $\zeta(s)$  is the Riemann zeta-function.

**Theorem 1.** *We have*

$$\#\mathcal{A}(N) \leq \frac{\pi(N)}{(\log_3 N)^{\vartheta+o(1)}}$$

as  $N \rightarrow \infty$ .

Our proof is based on an effective version of the Chebotarev Density Theorem that is due to Lagarias and Odlyzko [11].

## 2. PROOF OF THEOREM 1

**2.1. The strategy.** Observe that a nontrivial fixed point corresponds to a solution of the congruence

$$(8) \quad x^{x-1} \equiv 1 \pmod{p}, \quad x \in \{2, 3, \dots, p-1\}.$$

Thus, we wish to show that for almost all primes  $p$  the congruence (8) has a solution.

Given a prime  $p$  such that a “small” prime  $q$  divides  $p-1$ , we write  $p-1 = q \cdot a$ , so  $a = (p-1)/q$ . For an integer  $x$  of the form  $x = 1 + a\beta$ , with  $\beta \in \{1, \dots, q-1\}$ , we have

$$x^{x-1} \equiv (1 + a\beta)^{a\beta} \equiv (1 - \beta/q)^{\beta(p-1)/q} \pmod{p}.$$

Hence, we obtain a valid solution if  $1 - \beta/q \equiv (q - \beta)/q \pmod{p}$  is a  $q$ -th power modulo  $p$  for some  $0 < \beta < q$ . In other words, with  $n = q - \beta$ , we find that

$$x = 1 + a\beta = 1 + \frac{1}{q}(p-1)(n-q) \in [2, p-1]$$

is a solution to (8) provided that  $n/q$  is a  $q$ -th power modulo  $p$ . Thus, it suffices to show that there exists a  $q$ -th power modulo  $p$  of the form  $n/q$  with  $n \in [1, q-1]$ .

Note that the chance of a random element in the finite field of  $p$  elements  $\mathbb{F}_p$  being a  $q$ -th power equals  $1/q$ . So, heuristically, assuming that the set of  $q$ -th powers has sufficiently random behavior, we can expect that the probability of this *not* happening is  $(1 - 1/q)^{q-1} = 1/e + o(1)$  as  $q \rightarrow \infty$ .

The strategy we adopt is thus to consider primes  $p \equiv 1 \pmod{q}$  for “many”, say  $k$ , “small” (but not “too small”) primes  $q$ ; the “probability” that all such  $q$  fail to provide a valid solution  $x$  to the original congruence is expected to be about  $e^{-k}$ , *provided* that we can show that almost all primes  $p$  have such a property. We do this though not in a direct way. In particular, for the “individual” probability of  $q$  to fail we only obtain an upper bound of  $1 - \vartheta = 0.576\dots$  rather than  $1/e = 0.367\dots$

**2.2. The Chebotarev Density Theorem.** We let  $\mathbb{L}$  be a finite Galois extension of  $\mathbb{Q}$  with Galois group  $G$  of degree  $d = [\mathbb{L} : \mathbb{Q}]$  and discriminant  $\Delta$ . Let  $\mathcal{C}$  be a union of conjugacy classes of  $G$ . We define

$$\pi_{\mathcal{C}}(N, \mathbb{L}/\mathbb{Q}) = \#\{p \leq N : p \text{ unramified in } \mathbb{L}/\mathbb{Q}, \sigma_p \in \mathcal{C}\},$$

where  $\sigma_p$  is the Artin symbol of  $p$  in the extension  $\mathbb{L}/\mathbb{Q}$  (see [7]).

A combination of a version of the Chebotarev Density Theorem due to Lagarias and Odlyzko [11] with a bound of Stark [19] for a possible Siegel zero, yields the following result (see also [14, Lemma 6]).

**Lemma 2.** *There are absolute constants  $A_1, A_2 > 0$  such that if*

$$(9) \quad \log N \geq 10d(\log |\Delta|)^2$$

*then*

$$(10) \quad \left| \pi_{\mathcal{C}}(N, \mathbb{L}/\mathbb{Q}) - \frac{\#\mathcal{C}}{\#G} \text{li}(N) \right| \ll \frac{\#\mathcal{C}}{\#G} \text{li}(N^\beta) + \|\mathcal{C}\| N \exp\left(-A_1 \sqrt{\frac{\log N}{d}}\right)$$

*with some  $\beta$  satisfying the inequality*

$$\beta < 1 - \frac{A_2}{\max\{|\Delta|^{1/d}, \log |\Delta|\}},$$

*where  $\|\mathcal{C}\|$  is the number of conjugacy classes in  $\mathcal{C}$ .*

**2.3. Some preliminaries on Kummer extensions.** Let  $q$  be prime. We note that

$$\{p \leq N : p \equiv 1 \pmod{q}, n/q \text{ is a } q\text{-th power modulo } p\}$$

is, apart from the  $O(\log(qn))$  ramified primes all dividing  $qn$ , equal to the set of primes  $p \leq N$  such that  $p$  splits completely in the Kummer extension  $\mathbb{K}_{q,n} = \mathbb{L}_q(\sqrt[q]{n/q})$ , where  $\mathbb{L}_q = \mathbb{Q}(\zeta_q)$  is the cyclotomic extension generated by the primitive  $q$ -th root of unity  $\zeta_q = e^{2\pi i/q}$ . Note further that the condition that  $p$  splits completely in  $\mathbb{L}_q$  is equivalent to  $p \equiv 1 \pmod{q}$ .

The ideas behind our argument can be outlined as follows. Note that choosing a prime ideal  $P \mid p$  in the ring of integers of  $\mathbb{L}_q$  essentially amounts to choosing a nontrivial  $q$ -th root of unity in  $\mathbb{F}_p$ . Moreover, having made such a choice, the action of the Artin map  $\sigma_{P,n} \in \text{Gal}(\mathbb{K}_{q,n}/\mathbb{L}_q)$  (note that this Galois group is abelian) allows us, via Kummer theory, to associate with an integer  $n$  a canonical element in  $\mathbb{Z}/q\mathbb{Z}$ ; furthermore, this allows us to make “compatible” choices of elements in  $\mathbb{Z}/q\mathbb{Z}$  associated with different integers  $n$ .

To fix the ideas, let  $g$  be a nontrivial  $q$ -th root modulo  $p$ . By Kummer theory, we can then find “compatible” integers  $x_0, x_1, x_2, \dots, x_{q-1}$  modulo  $q$  such that  $g^{x_0} \in q \cdot (\mathbb{F}_p^\times)^q$ , and  $g^{x_n} \in n \cdot (\mathbb{F}_p^\times)^q$  for  $n = 1, 2, \dots, q-1$  (where  $(\mathbb{F}_p^\times)^q$  is set of  $q$ -th powers in  $\mathbb{F}_p^\times$  and  $\lambda \cdot (\mathbb{F}_p^\times)^q$  denotes the element-wise multiplication).

Note that knowledge of  $x_k$  for all *prime*  $k < q$ , determines  $x_n$  modulo  $q$  for  $n$  *composite*. Moreover, the condition that  $n/q$  is not a  $q$ -th power for all  $n \in [1, q-1]$  is equivalent to  $x_n \not\equiv x_0 \pmod{q}$  for  $1 \leq n \leq q-1$ .

**2.4. A system of linear forms modulo  $q$ .** Motivated by the arguments of Section 2.3, we study a system of certain linear equations modulo  $q$ . Let  $d = \pi(q-1)$ , and given an integer  $n \in [1, q-1]$ , define a linear form  $\mathcal{L}_n : \mathbb{F}_q^d \rightarrow \mathbb{F}_q$  by

$$\mathcal{L}_n(\mathbf{v}) = \sum_{i=1}^d \alpha_{i,n} v_i,$$

where  $\mathbf{v} = (v_1, \dots, v_d)$  and the coefficients  $\{\alpha_{i,n}\}$  are read from the prime factorization

$$n = \prod_{i=1}^d p_i^{\alpha_{i,n}}.$$

Given  $x_0 \in \mathbb{F}_q$ , we study

$$N_q = \#\{\mathbf{v} \in \mathbb{F}_q^d : \mathcal{L}_n(\mathbf{v}) \neq x_0 \text{ for all } n \in \{1, 2, \dots, q-1\}\}.$$

For  $q$  large, it seems reasonable to expect that  $N_q$  should be of size  $q^d/e$  since, for  $\mathbf{v}$  a fixed nonzero vector, the “probability” that  $\mathcal{L}_n(\mathbf{v}) \neq x_0$  for all  $n$  *if the forms are randomly chosen*, equals  $(1-1/q)^{q-1} \simeq 1/e$ . Equivalently, if we define

$$c(q) = N_q/q^d,$$

we expect that  $c(q) = 1/e + o(1)$  as  $q \rightarrow \infty$ .

While we are not able to prove that  $c(q)$  approaches  $1/e$  as  $q$  becomes large, we prove a weaker upper bound which is sufficient for our purposes.

**Lemma 3.** *As  $q$  tends to infinity, we have*

$$c(q) \leq 1 - \vartheta + o(1),$$

where  $\vartheta$  is given by (7).

*Proof.* For  $n > 1$ , the linear form  $\mathcal{L}_n$  is nontrivial and the equation  $\mathcal{L}_n(\mathbf{v}) = x_0$  has at least one solution; hence exactly  $q^{d-1}$  solutions. Further, given two *square-free* integers  $2 \leq m < n < q$ , we note that the corresponding linear forms  $\mathcal{L}_n$  and  $\mathcal{L}_m$  are independent. Thus, there are exactly  $q^{d-2}$  solutions  $\mathbf{v}$  to

$$\mathcal{L}_n(\mathbf{v}) = \mathcal{L}_m(\mathbf{v}) = x_0.$$

Let  $M$  denote the number of square-free positive integers up to  $q$ . Thus, we have  $M = (1/\zeta(2) + o(1))q$  as  $q \rightarrow \infty$ .

To obtain an upper bound, we discard the condition that  $\mathcal{L}_n(\mathbf{v}) \neq x_0$  for squarefull  $n$ . Then, removing those  $\mathbf{v}$  for which  $\mathcal{L}_n(\mathbf{v}) = x_0$  for some square-free  $n$ , and adding back in  $\mathbf{v}$ 's for which  $\mathcal{L}_n(\mathbf{v}) = \mathcal{L}_m(\mathbf{v}) = x_0$  for pairs of distinct square-free  $m, n$  (in essence, truncating the inclusion-exclusion principle at the third step), we find that

$$N_q \leq q^d - Mq^{d-1} + \binom{M}{2}q^{d-2} = q^d(1 - 1/\zeta(2) + 1/(2\zeta(2)^2) + o(1))$$

as  $q \rightarrow \infty$ , and the result follows.  $\square$

**2.5. Independence of field extensions.** For a prime  $q \mid Q$  we consider the algebraic number field

$$\mathbb{K}_q = \mathbb{Q}(\zeta_q, \sqrt[q]{2}, \sqrt[q]{3}, \sqrt[q]{5}, \dots, \sqrt[q]{q});$$

that is, we adjoin the  $q$ -th roots of the unity and the  $q$ -th roots of the primes  $p \leq q$  to  $\mathbb{Q}$ .

Assume that  $Q$  is a product of  $k$  distinct primes  $q_1, \dots, q_k$ . We define

$$\mathbb{K}_Q = \mathbb{K}_{q_1} \circ \mathbb{K}_{q_2} \circ \dots \circ \mathbb{K}_{q_k}$$

to be the composite field obtained from the fields  $\mathbb{K}_q$  as  $q$  ranges over the prime divisors of  $Q$ .

**Lemma 4.** *Assume that  $Q$  is an odd integer. Then the field extensions  $\mathbb{L}_q(\sqrt[q]{\ell})/\mathbb{L}_q$  are linearly disjoint as  $(q, \ell)$  ranges over pairs of primes such that  $\ell \leq q$  and  $q \mid Q$ .*

*Proof.* We break the argument in two steps.

First we show that if  $q$  is fixed, then  $\mathbb{L}_q(\sqrt[q]{\ell})/\mathbb{L}_q$  are linearly disjoint once  $\ell$  ranges over primes  $\ell \leq q$ . If this is not so, then there exist  $s \geq 2$  primes  $\ell_1, \dots, \ell_s$  such that  $\mathbb{L}_q \subsetneq \mathbb{K}$  where

$$\mathbb{K} = \mathbb{L}_q(\sqrt[q]{\ell_1}, \dots, \sqrt[q]{\ell_{s-1}}) \cap \mathbb{L}_q(\sqrt[q]{\ell_s}).$$

Observe that  $\mathbb{K}/\mathbb{Q}$  is normal as an intersection of normal extensions. We show that  $\mathbb{K} = \mathbb{L}_q(\sqrt[q]{\ell_s})$ . Indeed, if this is not so, then, by Galois theory, the group  $\text{Gal}(\mathbb{L}_q(\sqrt[q]{\ell_s})/\mathbb{K})$  is a proper nontrivial normal subgroup of  $\text{Gal}(\mathbb{L}_q(\sqrt[q]{\ell_s})/\mathbb{L}_q)$ , but this last group has order  $q$ , a prime number. This shows that  $\mathbb{K} = \mathbb{L}_q(\sqrt[q]{\ell_s})$ . So,

$$(11) \quad \mathbb{L}_q(\sqrt[q]{\ell_s}) \subseteq \mathbb{L}_q(\sqrt[q]{\ell_1}, \dots, \sqrt[q]{\ell_{s-1}}).$$

The discriminant of the field on the left is divisible only by the primes  $q$  and  $\ell_s$ , while the discriminant of the field on the right is divisible by the primes  $q$  and  $\ell_1, \dots, \ell_{s-1}$ . We get an immediate contradiction



unless  $\ell_s = q$ . So, it remains to treat the case  $\ell_s = q$ . If  $s = 2$ , then we get

$$\mathbb{L}_q(\sqrt[q]{q}) \subseteq \mathbb{L}_q(\sqrt[q]{\ell_1}).$$

Since both extensions above have the same degree  $q(q-1)$  over  $\mathbb{Q}$ , it follows that the above containment is in fact an equality. This is false because  $\ell_1$  ramifies in the field on the right but not in the field on the left.

Assume now that  $s \geq 3$  is minimal such that containment (11) holds for some prime  $q = \ell_s$  and some primes  $\ell_1 < \dots < \ell_{s-1} < q$ . Further, by the minimality of  $s$ ,  $\sqrt[q]{q}$  cannot belong to any field of the type  $\mathbb{Q}(\zeta_q, \sqrt[q]{\ell_{i_1}}, \dots, \sqrt[q]{\ell_{i_t}})$  for some proper subset  $\{i_1, \dots, i_t\}$  of  $\{1, \dots, s-1\}$ . Thus, we get a relation of the type

$$\sqrt[q]{q} = R_0 + R_1 \sqrt[q]{\ell_{s-1}} + \dots + R_{q-1} (\sqrt[q]{\ell_{s-1}})^{q-1},$$

where  $R_i = S_i(\zeta_q, \sqrt[q]{\ell_1}, \dots, \sqrt[q]{\ell_{s-2}})$  for some

$$S_i(X_0, X_1, \dots, X_{s-2}) \in \mathbb{Q}[X_1, \dots, X_{s-2}]$$

and at least one of  $R_1, \dots, R_{q-1}$  is nonzero. Hence,  $\sqrt[q]{\ell_{s-1}}$  is an algebraic number of degree at most  $q-1$  over the normal field

$$\mathbb{Q}(\zeta_q, \sqrt[q]{q}, \sqrt[q]{\ell_1}, \dots, \sqrt[q]{\ell_{s-2}}).$$

Since  $\mathbb{Q}(\ell_s^{1/q})$  is in fact of prime degree  $q$  over  $\mathbb{Q}$ , we get that

$$\sqrt[q]{\ell_{s-1}} \in \mathbb{Q}(\zeta_q, \sqrt[q]{q}, \sqrt[q]{\ell_1}, \dots, \sqrt[q]{\ell_{s-2}}),$$

giving

$$\mathbb{Q}(\sqrt[q]{\ell_{s-1}}) \subseteq \mathbb{Q}(\zeta_q, \sqrt[q]{q}, \sqrt[q]{\ell_1}, \dots, \sqrt[q]{\ell_{s-2}}).$$

However, this last field inclusion is false because the discriminant of the field on the left is divisible by the prime  $\ell_{s-1}$ , while the discriminant of the field on the right is divisible only by primes  $\ell_1, \dots, \ell_{s-2}$  and  $q$ .

We next show that the fields  $\mathbb{K}_q$  are linearly disjoint as  $q$  varies over the prime factors of  $Q$ . Again assume that this is not so and conclude that there exist  $s \geq 2$  prime factors of  $Q$  denoted  $q_1 < \dots < q_s$  such that

$$\mathbb{Q} \subset \mathbb{K} = \mathbb{K}_{q_1} \dots \mathbb{K}_{q_{s-1}} \cap \mathbb{K}_{q_s}.$$

Observe that all prime factors dividing the order of the Galois group of  $\mathbb{K}_{q_s}/\mathbb{Q}$  divide  $q_s(q_s-1)$ , while the Galois group of  $\mathbb{K}_{q_1} \dots \mathbb{K}_{q_{s-1}}$  has order divisible only by primes dividing  $q_1(q_1-1) \dots q_{s-1}(q_{s-1}-1)$ . Thus, the order of the Galois group  $\text{Gal}(\mathbb{K}/\mathbb{Q})$ , as a factor group of  $\text{Gal}(\mathbb{K}_{q_s}/\mathbb{Q})$ , can be divisible only by primes dividing  $q_s-1$ .

The subgroup  $\text{Gal}(\mathbb{K}_{q_s}/\mathbb{K})$  is normal, so by the above observation on possible prime divisors of its order, must contain the  $q_s$ -Sylow subgroup of  $\text{Gal}(\mathbb{K}_{q_s}/\mathbb{Q})$ , which is isomorphic to  $(\mathbb{Z}/q_s\mathbb{Z})^{\pi(q_s)}$ . However, the Galois group  $\text{Gal}(\mathbb{K}_{q_s}/\mathbb{Q})$  is isomorphic to a semidirect product of  $\mathbb{Z}/(q_s - 1)\mathbb{Z}$  with  $(\mathbb{Z}/q_s\mathbb{Z})^{\pi(q_s)}$ , where the first cyclic group acts diagonally as the group of automorphisms of  $\mathbb{Z}/q_s\mathbb{Z}$ . It is not hard to see that in the Galois group  $\text{Gal}(\mathbb{K}_{q_s}/\mathbb{Q})$ , the  $q_s$ -Sylow subgroup is maximal normal. This shows, via Galois correspondence between subgroups and subfields, that  $\text{Gal}(\mathbb{K}_{q_s}/\mathbb{K})$  is the  $q_s$ -Sylow subgroup, so  $\mathbb{K} = \mathbb{L}_{q_s}$  is the cyclotomic field.

In particular,  $\mathbb{K}$  contains  $q_s$ -th roots of unity and hence the discriminant of  $\mathbb{K}$  is divisible by  $q_s$  — a contradiction since the discriminant of  $\mathbb{K}_{q_1} \cdots \mathbb{K}_{q_{s-1}}$  is divisible only by primes up to  $q_{s-1}$ .

Altogether, this shows that the field extensions  $\mathbb{L}_q(\sqrt[q]{\ell})/\mathbb{L}_q$  are indeed linearly disjoint as  $(q, \ell)$  ranges over pairs of primes such that  $\ell \leq q$ , thereby concluding the proof.  $\square$

**2.6. Estimating the degree and discriminant of  $\mathbb{K}_Q$ .** We keep the notations from Section 2.5. Put  $d_Q$  and  $\Delta_Q$  for the degree and discriminant of  $\mathbb{K}_Q$ , respectively.

**Lemma 5.** *The bounds*

- (i)  $d_Q \leq \exp(q_k^2)$ ;
- (ii)  $\Delta_Q \leq \exp(\exp(2q_k^2))$

*hold for large enough  $k$ .*

*Proof.* It is clear that  $\mathbb{K}_Q$  is the compositum of

$$(12) \quad n = (\pi(q_1) + 1) + (\pi(q_2) + 1) + \cdots + (\pi(q_k) + 1) < \frac{q_k^2}{\log q_k}$$

fields  $\mathbb{K}_{i,j} = \mathbb{Q}(r_i^{1/q_j})$ , where  $r_i \in \{1\} \cup \{p \leq q_j\}$  and  $j = 1, \dots, k$ , each of degree at most  $q_k$ . The inequality (12) above holds for large  $k$ . Thus, (i) follows. For (ii), observe that the discriminant of each of  $\mathbb{K}_{i,j}$  is at most  $q_k^{2q_k}$ . Label these fields in some way as  $\mathbb{K}_1, \dots, \mathbb{K}_n$  and let  $\mathbb{L}_j = \mathbb{K}_1 \circ \mathbb{K}_2 \circ \cdots \circ \mathbb{K}_j$  for  $j = 1, \dots, n$ . Note that  $\mathbb{L}_{j+1} = \mathbb{L}_j \circ \mathbb{K}_{j+1}$ , therefore

$$\Delta_{\mathbb{L}_{j+1}} \leq \Delta_{\mathbb{L}_j}^{[\mathbb{K}_{j+1}:\mathbb{Q}]} \cdot \Delta_{\mathbb{K}_{j+1}}^{[\mathbb{L}_j:\mathbb{Q}]}.$$

Since  $[\mathbb{K}_{j+1} : \mathbb{Q}] \leq q_k$ ,  $[\mathbb{L}_j : \mathbb{Q}] \leq q_k^j$  and  $\Delta_{\mathbb{K}_j} \leq q_k^{2q_k}$ , we conclude that if we put  $\lambda_j$  for some constant such that  $\Delta_{\mathbb{L}_j} \leq q_k^{\lambda_j q_k^j}$ , then the inequalities

$$\lambda_1 \leq 2 \quad \text{and} \quad \lambda_{j+1} \leq \lambda_j + 2$$

hold for  $j = 1, \dots, n-1$ . Hence,  $\lambda_j \leq 2j$  for  $j = 1, \dots, n$ . With  $j = n$ , we obtain

$$\Delta_Q \leq q_k^{2nq_k^n} < q_k^{2q_k^{q_k^2+2}} < \exp(\exp(2q_k^2))$$

for all large  $k$ , thus proving (ii).  $\square$

**2.7. Some technical estimates.** For a square-free integer  $S$ , we define

$$c(S) = \prod_{q|S} c(q).$$

For positive integers  $L$  and  $R$  with  $Q = LR$ , define

$$\mathcal{P}_{L,R}(N) = \#\{p \leq N : \gcd(p-1, Q) = L\},$$

and

$$\tilde{\mathcal{P}}_{L,R}(N) = \{p \in \mathcal{P}_{L,R}(N) : n/q \notin (\mathbb{F}_p^\times)^q \text{ for all } q \mid L \text{ and } 0 < n < q\}.$$

**Lemma 6.** *If*

$$(13) \quad 6q_k^2 < \log_2 N,$$

*then*

$$(14) \quad \tilde{P}_{L,R}(N) \ll \pi(N) \cdot \frac{c(L)}{\varphi(L)} \cdot \prod_{q \mid R} \left( \frac{q-2}{q-1} \right).$$

*wh*

*Proof.* This follows from the Chebotarev density theorem. More precisely, a prime  $p$  counted by  $\tilde{P}_{L,R}(N)$  has the following property:  $p \equiv 1 \pmod{q}$  for each prime  $q \mid L$  and for all  $1 \leq n < q$ ,  $n/q$  is not a  $q$ -th power in  $\mathbb{F}_p^\times$ . In terms of the image of the Frobenius map, the relative size of the corresponding conjugacy classes in  $\text{Gal}(\mathbb{K}_q/\mathbb{Q})$ , is given by  $c(q)$  (see Section 2.4). Since by Lemma 4 the field extensions  $\mathbb{K}_{q_i}$  are linearly disjoint for  $i = 1, \dots, k$ , the relative size inside  $\text{Gal}(\mathbb{K}_L/\mathbb{Q})$  is given by  $c(L)$ . This takes care of the main term. For the error term, we appeal to Lemmas 2 and 5. More precisely, by Lemma 5, we have

$$10d_Q(\log \Delta_Q)^2 < 10\exp(5q_k^2) < \log N$$

for large  $k$  by the assumption (13), so the inequality (9) holds. As for error terms, we have

$$d_Q \leq \exp(q_k^2) < (\log N)^{1/6}$$

so the second error term in (10) is negligible with respect to the main term. Finally, we note that the first error term in (10) is at most comparable with the main term and it could be incorporated into it given that (14) is only an upper bound estimate.  $\square$

We now set

$$(15) \quad Q_t = \prod_{t < q \leq e^t} q.$$

Thus,  $Q$  has  $k = \pi(e^t) - \pi(t)$  prime factors labeled  $q_1, \dots, q_k$ . The inequality (13) is satisfied for this choice of  $Q$  provided that  $N$  is large and

$$(16) \quad t = \frac{1}{3} \log_3 N.$$

We get the following result.

**Lemma 7.** *If  $N$  is large and (16) holds, then*

$$P_{1, Q_t}(N) \ll \frac{\pi(N) \log_4 N}{\log_3 N}.$$

*Proof.* By the Brun sieve [20, Theorem 3, Section I.4.2], and on recalling Mertens formula [20, Section I.1.5], we have

$$\#\{p \leq N : \gcd(p-1, Q_t) = 1\} \ll \pi(N) \prod_{q|Q_t} \left( \frac{q-2}{q-1} \right) \ll \frac{\pi(N) \log t}{t},$$

and the result now follows from (16).  $\square$

**2.8. Concluding the proof.** We assume that  $Q_t$  is given by (15) where  $t$  is given by (16). In particular, the conditions of Lemmas 6 and 7 are satisfied.

By Lemma 6, we have

$$(17) \quad \sum_{LR=Q_t, L>1} \tilde{P}_{L,R}(N) \ll \pi(N) \sum_{LR=Q_t, L>1} \frac{c(L)}{\varphi(L)} \cdot \prod_{q|R} \left( \frac{q-2}{q-1} \right).$$

Furthermore,

$$\begin{aligned} \sum_{\substack{LR=Q_t \\ L>1}} \frac{c(L)}{\varphi(L)} \cdot \prod_{q|R} \left( \frac{q-2}{q-1} \right) &= \prod_{q|Q_t} \left( \frac{q-2}{q-1} \right) \sum_{\substack{LR=Q_t \\ L>1}} \frac{c(L)}{\varphi(L)} \cdot \prod_{q|L} \frac{q-1}{q-2} \\ &= \prod_{q|Q_t} \left( \frac{q-2}{q-1} \right) \sum_{\substack{LR=Q_t \\ L>1}} c(L) \cdot \prod_{q|L} \frac{1}{q-2} \\ &\leq \prod_{q|Q_t} \left( \frac{q-2}{q-1} \right) \sum_{L|Q_t} \prod_{q|L} \left( \frac{c(q)}{q-2} \right) \\ &= \prod_{q|Q_t} \left( \frac{q-2}{q-1} \right) \prod_{q|Q_t} \left( 1 + \frac{c(q)}{q-2} \right) = \prod_{q|Q_t} \left( 1 - \frac{1-c(q)}{q-1} \right). \end{aligned}$$

Thus, recalling (17), we obtain

$$\sum_{LR=Q_t, L>1} \tilde{P}_{L,R}(N) \ll \pi(N) \prod_{q|Q_t} \left(1 - \frac{1-c(q)}{q-1}\right).$$

Using Lemma 3 and then the Mertens formula again, we obtain

$$\begin{aligned} \prod_{q|Q_t} \left(1 - \frac{1-c(q)}{q-1}\right) &\ll \exp \left( - \sum_{q|Q_t} \frac{1-c(q)}{q} \right) \\ &\ll \exp \left( -(\vartheta + o(1)) \sum_{q|Q_t} \frac{1}{q} \right) \\ &= \exp \left( -(\vartheta + o(1)) \log t \right) = \frac{1}{(\log_3 N)^{\vartheta+o(1)}}, \end{aligned}$$

and so

$$\sum_{LR=Q_t, L>1} \tilde{P}_{L,R}(N) \leq \frac{\pi(N)}{(\log_3 N)^{\vartheta+o(1)}}$$

as  $N \rightarrow \infty$ . With Lemma 7, we finally get that

$$\begin{aligned} \#\mathcal{A}(N) &\ll P_{1,Q_t}(N) + \sum_{LR=Q_t, L>1} \tilde{P}_{L,R}(N) \\ &\ll \pi(N) \left( \frac{\log_4 N}{\log_3 N} + \frac{1}{(\log_3 N)^{\vartheta+o(1)}} \right), \end{aligned}$$

as  $N \rightarrow \infty$ , which finishes the proof.

### 3. FURTHER REMARKS ON $\#\mathcal{A}(N)$

**3.1. Heuristic arguments.** Recall that  $x = 1$  is always a trivial fixed point, and note that  $x = p - 1$  is never a fixed point. Hence, we only consider  $x$  whose multiplicative order is greater than two, and the exponent  $x - 1$  ranging over integers in the interval  $[1, p - 3]$ .

If  $d|p - 1$  and  $x$  is a primitive  $d$ -th root of unity *and* we make the assumption that the exponent  $x - 1$  is “independent” of  $x$ , the “chance” that  $x^{x-1} \equiv 1 \pmod{p}$  equals the chance that  $d|x - 1$ ; this occurs with probability

$$(18) \quad \frac{\lfloor (p-3)/d \rfloor}{p-3} = \frac{(p-1)/d - 1}{p-3} = 1/d + O(1/p).$$

Letting  $x$  range over the set of  $\varphi(d)$  primitive  $d$ -th roots of unity, the probability that  $x^{x-1} \not\equiv 1 \pmod{p}$  for all of them, assuming independence, equals  $\left(1 - \frac{\lfloor (p-3)/d \rfloor}{p-3}\right)^{\varphi(d)}$ . Moreover, with the further assumption of independence when  $d$  ranges over divisors of  $p-1$ , this suggests that

$$\#\mathcal{A}(N) = (1 + o(1))H(N)$$

as  $N \rightarrow \infty$ , where

$$(19) \quad H(N) = \sum_{p < N} \prod_{\substack{d|p-1 \\ 2 < d < p-1}} \left(1 - \frac{\lfloor (p-3)/d \rfloor}{p-3}\right)^{\varphi(d)}.$$

For  $p$  fixed (but large) we similarly find that the heuristic probability of the map  $\psi_p$  having no (nontrivial) fixed points, using that

$$\left(1 - \left(\frac{1}{d} + O(p^{-1})\right)\right)^{\varphi(d)} = \exp\left(\varphi(d) \ln\left(1 - \left(\frac{1}{d} + O(p^{-1})\right)\right)\right).$$

is given by  $\exp(-\Delta_p)$ , where

$$\begin{aligned} \Delta_p &:= - \sum_{\substack{d|p-1 \\ 2 < d < p-1}} \varphi(d) \ln\left(1 - \left(\frac{1}{d} + O(p^{-1})\right)\right) \\ &= \sum_{\substack{d|p-1 \\ 2 < d < p-1}} \varphi(d) \left(\frac{1}{d} + \frac{1}{2d^2} + O(p^{-1} + d^{-3})\right) \\ (20) \quad &= \sum_{d|p-1} \varphi(d) \left(\frac{1}{d} + \frac{1}{2d^2} + O(p^{-1} + d^{-3})\right) + O(1) \\ &= \tau(p-1) \cdot \prod_{q^e || p-1} \left(1 - \frac{e}{(1+e)q}\right) + O\left(\sum_{d|p-1} 1/d\right). \end{aligned}$$

Hence,  $\psi_p$  is exceeding likely to have a nontrivial fixed point unless  $p-1$  have rather few prime factors. Restricting to  $p$  such that  $p-1$  is square-free, and, motivated by the results of Sathe [16] and Selberg [17], assuming that for any fixed  $\varepsilon > 0$  and  $k \leq (2 - \varepsilon) \log_2 N$ , we have

$$\#\{p \leq N : \omega(p-1) = k\} \sim \frac{N(\log_2 N)^{k-1}}{(k-1)! \log^2 N}$$

we expect that the number of  $p \leq x$  such that  $\psi_p$  has no nontrivial fixed point modulo  $p$  is, for any integer  $k > 0$ , is

$$H(N) \gg \sum_{p \leq N} \exp(-\Delta_p) \geq \sum_{1 \leq k \leq (2-\varepsilon) \log_2 N} \frac{N(\log_2 N)^{k-1} \exp(-2^{k+o(k)})}{(k-1)! \log^2 N}.$$

Using the trivial estimate  $1 \leq (k-1)! \leq k^k$  we see that  $(k-1)!$  can be absorbed in  $2^{k+o(k)}$  in the exponent. Furthermore, for any positive integer  $k \leq (2-\varepsilon) \log_2 N$  we have

$$H(N) \gg \frac{N \exp(k \log_3 N - 2^{k+o(k)})}{(\log N)^2 \log_2 N}.$$

Thus, taking

$$k = \left\lfloor \left( \frac{1}{\ln 2} - \eta \right) \log_4 N \right\rfloor,$$

for an arbitrary  $\eta > 0$  gives the bound

$$H(N) \geq \frac{N}{(\log N)^2} \exp((1/\ln 2 - \eta + o(1)) \log_3 N \log_4 N)$$

(note that using other admissible values of  $k$  does not significantly improve this bound; just one optimally chosen value suffices.) Since  $\eta > 0$  is arbitrary, we obtain the expected lower bound (6).

In fact we believe that the lower bound (6) is close to the actual order of magnitude of both  $\#\mathcal{A}(N)$  and  $H(N)$ .

The above argument, in particular (18), also suggests that the expected value of the total number of nontrivial fixed points over all primes  $p \leq N$  is

$$\sum_{p \leq N} F(p) = (1 + o(1))K(N)$$

where

$$(21) \quad K(N) = \sum_{p \leq N} \sum_{\substack{d|p-1 \\ d > 2}} \frac{\varphi(d)}{d} = \sum_{d=3}^N \frac{\varphi(d)}{d} \sum_{\substack{p \leq N \\ p \equiv 1 \pmod{d}}} 1.$$

Using the approximation

$$\sum_{\substack{p \leq N \\ p \equiv 1 \pmod{d}}} 1 = (1 + o(1)) \frac{N}{\varphi(d) \log N},$$

it seems reasonable to expect that

$$K(N) = (1 + o(1))N.$$

**3.2. Numerical results.** In Table 1 we compare the observed data for all primes  $p \leq N$  for  $N = 100000 \cdot k$ ,  $1 \leq k \leq 10$ , that have no nontrivial fixed point with the heuristically predicted value  $H(N)$  given by (19).

$N$	Observed	Predicted	Relative error
100000	567	585.6	-0.0318
200000	1007	1020.6	-0.0134
300000	1358	1421.4	-0.0446
400000	1715	1790.1	-0.0419
500000	2068	2151.8	-0.0389
600000	2404	2490.0	-0.0345
700000	2725	2826.7	-0.0360
800000	3053	3151.0	-0.0311
900000	3350	3479.5	-0.0372
1000000	3632	3796.2	-0.0433

TABLE 1. Number of primes  $p \leq N$  with no nontrivial fixed point

In Table 2 we present data for the total number of fixed points for all primes  $p \leq N$  for  $N = 50000 \cdot k$ ,  $1 \leq k \leq 9$ , that have no nontrivial fixed point, and compare it with the heuristically predicted value given by (21).

$N$	Observed	Predicted	Relative error
500000	465413	410686.1	0.1333
1000000	936280	831872.7	0.1255
1500000	1408964	1256499.5	0.1213
2000000	1883411	1683081.9	0.1190
2500000	2357781	2110954.9	0.1169
3000000	2832933	2539862.9	0.1154
3500000	3306597	2968852.5	0.1138
4000000	3780495	3398836.9	0.1123
4500000	4256757	3829903.3	0.1115

TABLE 2. Total number of observed nontrivial fixed points for  $p \leq N$  vs. random model prediction.

When comparing predicted and observed values we note that there seems to be a consistent negative bias in Table 1 and a consistent positive bias in Table 2. As of now, we have no satisfactory explanation of this phenomenon.



4. REMARKS ON THE DYNAMICS OF THE MAP  $\psi_p$ 

**4.1. Orbit length model.** Given a finite set  $X$ , a map  $\eta : X \rightarrow X$ , and a starting point  $x_0$ , define  $x_{n+1} = \eta(x_n)$  for  $n \in \mathbb{Z}^+$ . Let  $O_{\eta, x_0}(X) = \{x_0, x_1, \dots\}$  denote the forward orbit of  $x_0$  under  $\eta$ . Clearly, we have the trivial inequality  $\#O_{\eta, x_0}(X) \leq \#X$ , but if  $\eta$  is a *random map* (that is, for each  $x \in X$ , we define its image  $\eta(x)$  by uniformly selecting a random element of  $X$ ), a simple ‘birthday paradox’ argument shows that  $\#O_{\eta, x_0}(X)$  is very likely to be of size roughly  $(\#X)^{1/2}$ ; in particular, as  $\#X \rightarrow \infty$ ,  $\#O_{\eta, x_0}(X) \leq (\#X)^{1/2+o(1)}$  holds with probability one.

Thus, if we naïvely model  $\psi_p$  as a random map, then, as  $p \rightarrow \infty$ , and selecting a random starting point  $x_0$ , the orbit size  $\#O_{\psi_p, x_0}(\mathbb{F}_p)$  is expected to be roughly of size  $\sqrt{p}$ , see [6]. However, numerics indicate that  $\#O_{\psi_p, x_0}(\mathbb{F}_p)$  often is *much smaller* than  $\sqrt{p}$ . In fact, in what follows, we give numerical evidence, and an heuristic model, that the probability density distribution of  $\log \#O_{\psi_p, x_0}(\mathbb{F}_p) / \log p$  has support in  $[0, 1/2]$ .

In fact, it is easy to see that the orbit  $O_{\psi_p, x_0}(\mathbb{F}_p)$  are shorter than expected from a random map as once a certain element  $x \in O_{\psi_p, x_0}(\mathbb{F}_p)$  lies in a multiplicative subgroup of  $\mathbb{F}_p^*$ , then so does  $\psi_p(x)$ , and the remaining part of the orbit never leaves this subgroup. So, the behavior of orbits of  $\psi_p$ , originating at a point  $x_0 \in \mathbb{F}_p^*$  is ruled by two (apparently independent) factors:

- random map-like behavior inside of a subgroup of  $\mathbb{F}_p^*$  which eventually leads to a cycle formed by the ‘birthday paradox’ (see [6] for an exhaustive treatise of the structure of random maps);
- reducing the size of the multiplicative subgroup where the iterations of  $\psi_p$  get locked in as they progress along the trajectory.

For example, if the initial point  $x_0$  is not a primitive root of  $\mathbb{F}_p$ , this immediately puts all elements of the corresponding trajectory in a nontrivial multiplicative subgroup of  $\mathbb{F}_p^*$ .

Hence, we believe that the main reason for such small orbit lengths is that a correct model for  $\psi_p$  is that of a *random automorphism* on  $C_{p-1}$ , the cyclic group of cardinality  $p-1$ . Since  $\psi_p$  maps  $\mathbb{F}_p^\times$  into itself, and, as groups  $\mathbb{F}_p^\times \simeq C_{p-1}$ , we may translate the dynamics  $x_0 \rightarrow x_1 \rightarrow \dots$  on  $\mathbb{F}_p^\times$  to dynamics  $y_0 \rightarrow y_1 \rightarrow \dots$  on  $C_{p-1}$ . Under the assumption that the discrete log map (which identifies  $\mathbb{F}_p^\times$  with  $C_{p-1}$ ) behaves randomly, the image of  $\psi_p$  as a map of  $C_{p-1}$  be viewed as

“random” map  $\varphi : C_{p-1} \rightarrow C_{p-1}$  given by

$$\varphi(y) \equiv \alpha_y y \pmod{p-1},$$

where  $\alpha_y \in \mathbb{Z}/(p-1)\mathbb{Z}$  is selected randomly. In particular, once an iterate  $y_n$  “lands” in a subgroup  $H \subset C_{p-1}$ , it never “leaves”; and this makes much shorter orbit lengths likely.

For example, for primes  $p$  such that  $p-1 = s \cdot t$ , where  $s$  is the  $p^{1/3}$ -smooth part of  $p-1$ , and  $s \gg p^{1/3}$ , we find that it is very likely that the  $s$ -part of the orbit gets annihilated after at most  $p^{1/3+\varepsilon}$  steps (write  $C_{p-1} \simeq C_s \times C_t$  and say that the  $s$ -part of  $y_n$  is annihilated if the image of  $y_n$  in  $C_s \times C_t$  is of the form  $(0, *)$ .) In fact, if a prime  $q$  divides  $p-1$ , it is easy to see that the probability of the  $q$ -part not being annihilated after  $k$  steps is given by  $(1 - 1/q)^k$ , which, if  $q/k = o(1)$ , is  $o(1)$  as  $q \rightarrow \infty$ .

This leads to the following natural question. Let  $\Psi_{d,p}$  be the endomorphisms of  $\mathbb{F}_p^*$ , (indexed by the divisors  $d \in [1, p-1]$ ) and generated by the map  $x \mapsto x^d$ ,  $x \in \mathbb{F}_p^*$ .

**Question 8.** *Let  $x_0 \in \mathbb{F}_p^*$  be chosen uniformly at random and let  $\Psi_{d_1,p}, \dots, \Psi_{d_L,p}$  be a sequence of  $L$  random endomorphisms such that for every  $j = 1, \dots, L$  and  $d \mid p-1$  we have*

$$\Pr[(p-1, d_j) = d] = \frac{\varphi((p-1)/d)}{p-1},$$

*What is the expected size of the smallest subgroup of  $\mathbb{F}_p^*$  that contains the element  $\Psi_{d_L,p}(\dots(\Psi_{d_1,p}(x_0))\dots)$ ?*

Certainly, a version of Question 8 can be asked for any finite subgroup.

**4.2. Orbit length statistics.** If  $\eta$  behaves sufficiently randomly, then  $\#O_{\eta,x_0}(\mathbb{F}_p) \leq p^{1/2+o(1)}$  is very likely to hold. In fact, it is known that for  $\eta$  random,  $(\#O_{\eta,x_0}(\mathbb{F}_p))^2/(2p)$  converges in distribution to a mean one exponential as  $p \rightarrow \infty$ . In particular, the support of  $\log \#O_{\eta,x_0}(\mathbb{F}_p)/\log p$  is essentially concentrated around  $1/2$ .

See Figure 1 for an illustration of this well-known phenomenon, which also forms the basis of the so-called *Pollard’s rho-factorisation* algorithm, see [4, Section 5.2.1].

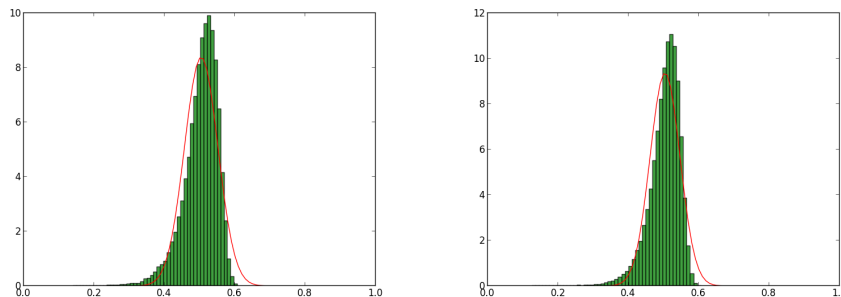


FIGURE 1. Histogram plot of  $\log \#O_{\eta, x_0}(\mathbb{F}_p) / \log p$  with  $\eta(x) = x^2 + 1$  for  $p \leq 1000000$  (left) and  $p \leq 5000000$  (right). Red curves indicate normal distributions with mean and variance fitted to the data.

However, the orbit sizes of  $\psi_p$  behaves very differently.

We remark that if  $p = 2q + 1$  where  $q$  is a Sophie Germain prime, then the second effect is negligible. Since the standard heuristic suggests a (relative) abundance of Sophie Germain primes, “on average” over primes  $p$ , the second effect is essentially invisible. However for a “typical” prime the situation is quite different. In other words, under the standard heuristic expectation of abundance of Sophie Germain prime, the average value of the trajectory length is of order  $p^{1/2}$  (possibly with some logarithmic factors), while the typical value is much smaller.

Furthermore, let  $P(k)$  denote the largest prime divisor of an integer  $k \geq 1$ . If  $\alpha \in (0, 1)$  and  $p$  runs through a sequence of primes with  $p - 1 = q \cdot s$  where  $q = P(p - 1) = p^{\alpha + o(1)}$  and  $s$  is  $p^{\alpha/2}$ -smooth (which conjecturally holds for a positive proportion of the primes for any  $\alpha \in (0, 1)$ ), we expect that a random endomorphism has the orbit of size at most  $p^{\alpha/2 + o(1)}$ . In turn, this suggests that the probability density function of  $\log \#O_{\psi_p, x_0}(\mathbb{F}_p) / \log p$  is supported in the full interval  $[0, 1/2]$ ; see Figure 2 for an illustration of this phenomenon.

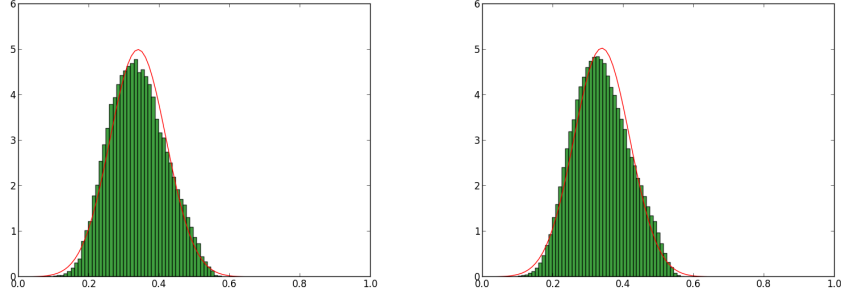


FIGURE 2. Histogram plots of  $\log \#O_{\psi_p, x_0}(\mathbb{F}_p) / \log p$ ,  $p \leq 10000000$  (left) and  $p \leq 50000000$  (right). Red curves indicate normal distributions with mean and variance fitted to the data.

To further show the difference in orbit statistics, it is also interesting to compare statistics when normalized by dividing by  $\sqrt{p}$ , see Figure 3.

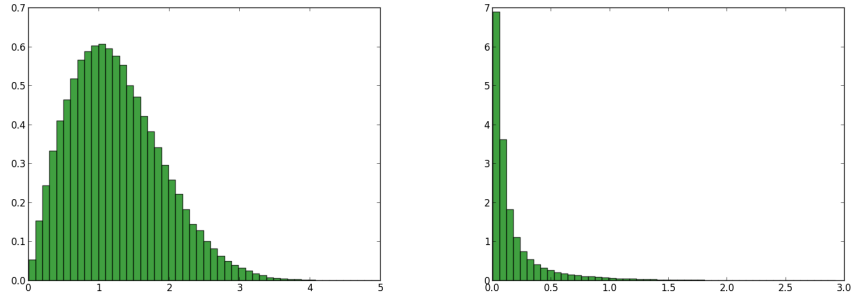


FIGURE 3. Histogram plots of  $\#O_{\eta, x_0}(\mathbb{F}_p) / \sqrt{p}$  with  $\eta(x) = x^2 + 1$  (left) and  $\#O_{\psi_p, x_0}(\mathbb{F}_p) / \sqrt{p}$  (right) for  $p \leq 50000000$ .

## 5. COMMENTS AND EXTENSIONS

As we have mentioned in Section 2.4, it is natural to expect that the following holds:

**Conjecture 9.** *Let  $x_0 \in \mathbb{F}_q$ . Then*

$$\frac{|\#\{\mathbf{v} \in \mathbb{F}_q^d : L_n(\mathbf{v}) \neq x_0 \text{ for } 1 \leq n \leq q\}|}{q^d} = e^{-1} + o(1),$$

as  $q \rightarrow \infty$ , where  $d = \pi(q - 1)$ .

In particular, Conjecture 9 implies that  $\vartheta \simeq 0.4231 \dots$  in the bound of Theorem 1 can be replaced with  $1 - 1/e \simeq 0.6321 \dots$ .

Clearly the map  $\psi_p$ , as any map over  $\mathbb{F}_p$  can be interpolated by polynomial, that is, for some unique polynomials  $F_p(X) \in \mathbb{F}_p[X]$  of degree at most  $p-1$  we have  $\psi_p(x) = F_p(x)$  for  $x \in \mathbb{F}_p$ . It is natural to use  $D_p = \deg F_p$  as a measure of “non-polynomiality” of the map  $\psi_p$ . In particular, we expect that  $D_p$  is close to its largest possible value  $p-1$ . Although we have not been able to establish this we show that

$$(22) \quad D_p \geq \left( \sqrt{2 - \sqrt{3}} + o(1) \right) p^{1/2} = 0.5176 \dots p^{1/2}.$$

We remark that the  $x^x$  is a quadratic non-residue modulo  $p$  if and only if both  $x$  is odd and a quadratic non-residue. Using the Pólya–Vinogradov bound of sums of quadratic characters, it is trivial to show that there are  $p/4 + O(p^{1/2} \log p)$  such values of  $x = 0, 1, \dots, p-1$ . Hence, for the sum of the Legendre symbols with  $F_p$  we have

$$\sum_{x \in \mathbb{F}_p} \left( \frac{F_p(x)}{p} \right) = p/2 + O(p^{1/2} \log p).$$

On the other hand, the results of Korobov [10] and Mit’kin [13] (which we use in a simplified form) imply that

$$\left| \sum_{x \in \mathbb{F}_p} \left( \frac{F_p(x)}{p} \right) \right| \leq D_p \sqrt{p - D_p^2/4 + O(D_p)}$$

(provided that, say,  $p \geq D_p^2/2 + 5$ ), which now implies (22).

For a prime  $p$  and a polynomial  $f(X) \in \mathbb{Z}[X]$  we denote by  $T_f(p)$  the number of solutions to the congruence

$$(23) \quad x^{f(x)} \equiv 1 \pmod{p}, \quad 1 \leq x \leq p-1.$$

We note that the number of fixed points of  $x \rightarrow x^{f(x)}$  is given by  $T_{f-1}(p)$ .

**Theorem 10.** *If  $f$  is squarefree, we have*

$$T_f(p) \leq p^{6/13 + o(1)}$$

as  $p \rightarrow \infty$ .

*Proof.* Let us fix  $d \mid p-1$  and denote by  $\mathcal{X}_d$  the set of solutions to (23) with

$$\gcd(f(x), p-1) = d.$$

Clearly any element  $x \in \mathcal{X}_d$  belongs to the multiplicative group  $\mathcal{G}_d \subseteq \mathbb{F}_p^*$  of index  $d$  in the multiplicative group  $\mathbb{F}_p^*$  of a finite field  $\mathbb{F}_p$  of  $p$  elements. Therefore,

$$(24) \quad \#\mathcal{X}_d \leq d.$$

Since  $f(X)$  is squarefree, by the Nagell–Ore theorem (see [9] for its strongest known form) for each  $d$  there is a set  $\mathcal{K}_d \subseteq \{0, \dots, d-1\}$  of cardinality  $\#\mathcal{K}_d = d^{o(1)}$  and such that every  $x \in \mathcal{X}_d$  satisfies

$$(25) \quad x \equiv k \pmod{d}$$

for some  $k \in \mathcal{K}_d$ . Let us fix  $k \in \mathcal{K}_d$  and denote by  $\mathcal{X}_{d,k}$  the set of  $x \in \mathcal{X}_d$  satisfying (25). Obviously,

$$(26) \quad \#\mathcal{X}_{d,k} \leq (p-1)/d.$$

Thus, in particular, from (24) and (26), we see that  $\#\mathcal{X}_{d,k} \leq \sqrt{p-1}$ . However, we now obtain a better bound.

We remark that the difference set

$$\mathcal{U}_{d,k} = \{x_1 - x_2 : x_1, x_2 \in \mathcal{X}_{d,k}\} \subseteq \mathbb{F}_p$$

is of cardinality at most

$$(27) \quad \#\mathcal{U}_{d,k} \leq 2(p-1)/d$$

as it is contained in the reductions modulo  $p$  of integers  $y \equiv 0 \pmod{d}$  from the interval  $y \in [-(p-1), p-1]$ . Similarly, for

$$\mathcal{W}_{d,k} = \{x_1 + x_2 - x_3 - x_4 : x_1, x_2, x_3, x_4 \in \mathcal{X}_{d,k}\} \subseteq \mathbb{F}_p,$$

we have

$$(28) \quad \#\mathcal{V}_{d,k} \leq 4(p-1)/d.$$

Furthermore, the product set

$$\mathcal{W}_{d,k} = \{x_1 x_2 : x_1, x_2 \in \mathcal{X}_{d,k}\} \subseteq \mathbb{F}_p$$

is of cardinality at most

$$(29) \quad \#\mathcal{W}_{d,k} \leq d$$

as it is contained in  $\mathcal{G}_d$ . Finally, as in [3, Section 1], we note that the Cauchy inequality implies that

$$E_{d,k} = \#\{(x_1, x_2, x_3, x_4) \in \mathcal{X}_{d,k}^4 : x_1 x_2 = x_3 x_4\}$$

satisfies

$$(30) \quad E_{d,k} \geq \frac{(\#\mathcal{X}_{d,k})^4}{\#\mathcal{W}_{d,k}}.$$

By the result of Bourgain and Garaev [3, Theorem 1.1] we have

$$E_{d,k}^4 \leq \left( \#\mathcal{U}_{d,k} + \frac{(\#\mathcal{X}_{d,k})^3}{p} \right) (\#\mathcal{X}_{d,k})^5 (\#\mathcal{U}_{d,k})^4 \#\mathcal{V}_{d,k} p^{o(1)},$$

which together with (30) implies

$$(31) \quad (\#\mathcal{X}_{d,k})^{11} \leq \left( \#\mathcal{U}_{d,k} + \frac{(\#\mathcal{X}_{d,k})^3}{p} \right) (\#\mathcal{U}_{d,k})^4 \#\mathcal{V}_{d,k} (\#\mathcal{W}_{d,k})^4 p^{o(1)}$$

as  $p \rightarrow \infty$ . Substituting the bounds (27), (28) and (29) in (31), we derive

$$(\#\mathcal{X}_{d,k})^{11} \leq \left( pd^{-1} + \frac{(\#\mathcal{X}_{d,k})^3}{p} \right) p^{5+o(1)} d^{-1}.$$

Thus,

$$(32) \quad \#\mathcal{X}_{d,k} \leq \max \{ p^{6/11} d^{-2/11}, p^{1/2} d^{-1/8} \} p^{o(1)}.$$

Using (24) for  $d < p^{6/13}$  and (24) for  $d \geq p^{6/13}$ , we obtain

$$\#\mathcal{X}_{d,k} \leq p^{6/13+o(1)},$$

as  $p \rightarrow \infty$ ), which concludes the proof.  $\square$

**Remark 11.** We note that as long as  $d$  is square free, we have  $\#\mathcal{K}_d = d^{o(1)}$  with no assumption of  $f$  being square free. Hence, we find that the upper bound on  $T_f(p)$  holds without any assumption on  $f(x)$  provided that  $p-1$  is square free. In fact, it is enough to assume that the square full part of  $p-1$  is of size  $p^{o(1)}$ .

**Remark 12.** It is quite possible that using the results and arguments of Rudnev [15] one can improve the bound of Theorem 10.

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## REFERENCES

- [1] A. Balog, K. A. Broughan and I. E. Shparlinski, ‘On the number of solutions of exponential congruences’, *Acta Arith.*, **148** (2011), 93–103.
- [2] A. Balog, K. A. Broughan and I. E. Shparlinski, ‘Sum-products estimates with several sets and applications’, *Integers*, **12** (2012), 895–906.
- [3] J. Bourgain and M. Z. Garaev, ‘On a variant of sum-product estimates and explicit exponential sum bounds in prime fields’, *Math. Proc. Cambr. Phil. Soc.*, **146** (2008), 1–21.
- [4] R. Crandall and C. Pomerance, *Prime numbers: A computational perspective*, Springer-Verlag, New York, 2005.
- [5] R. Crocker, ‘On residues of  $n^n$ ’, *Amer. Math. Monthly*, **76** (1969), 1028–1029.
- [6] P. Flajolet and A. M. Odlyzko, ‘Random mapping statistics’, *Lecture Notes in Comput. Sci.*, **434** (1990), 329–354.
- [7] G. Gras, *Class field theory*, Springer-Verlag, Berlin, 2005.
- [8] J. Holden and P. Moree, ‘Some heuristics and results for small cycles of the discrete logarithm’, *Math. Comp.*, **75** (2006), 419–449.
- [9] M. N. Huxley, ‘A note on polynomial congruences’, *Recent Progress in Analytic Number Theory, Vol.1*, Academic Press, 1981, 193–196.
- [10] N. M. Korobov, ‘An estimate of the sum of the Legendre symbols’, *Dokl. Akad. Nauk SSSR* **196** (1971), 764–767 (in Russian); translated in *Soviet Math. Dokl.*, **12** (1971), 241–245.
- [11] J. C. Lagarias and A. M. Odlyzko, ‘Effective versions of the Chebotarev density theorem’, *Algebraic Number Fields*, Academic Press, New York, 1977, 409–464.
- [12] A. J. Menezes, P. C. van Oorschot and S. A. Vanstone, *Handbook of applied cryptography*, CRC Press, Boca Raton, FL, 1996.
- [13] D. A. Mit’kin, ‘An estimate of the sum of the Legendre symbols with polynomials of an even degree’, *Matem. Zametki*, **14** (1973), 73–81 (in Russian); translated in *Math. Notes*, **14** 1973, 597–602.
- [14] C. Pomerance and I. E. Shparlinski, ‘Rank statistics for a family of elliptic curves over a function field’, *Pure and Applied Mathem. Quart.*, **6** (2010), 21–40.
- [15] M. Rudnev, ‘An improved sum-product inequality in fields of prime order’, *Intern. Math. Res. Notices*, **2012** (2012), Article rnr158, 3693–3705.
- [16] L. G. Sathe, ‘On a problem of Hardy and Ramanujan on the distribution of integers having a given number of prime factors’, *J. Indian Math. Soc.*, **17** (1953), 27–81.
- [17] A. Selberg, ‘Note on a paper of L. G. Sathe’, *J. Indian Math. Soc.*, **18** (1954), 83–87.
- [18] L. Somer, ‘The residues of  $n^n$  modulo  $p$ ’, *The Fibonacci Quart.*, **19** (1981), 110–117.
- [19] H. M. Stark, ‘Some effective cases of the Brauer-Siegel theorem’, *Invent. Math.*, **3** (1974), 135–152.
- [20] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge University Press, 1995.



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